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An Improved Shooting Method for a Class of Switching Optimal Control Problems

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1. Introduction

In this study, we consider a class of switching optimal control problems which control functions take values from a discrete set and appear linearly in the objective function and the dynamical equations. This type of optimal control problems arises in many applications such as aerospace engineering, cranes, and industrial robots. The computation of optimal controls in this type of optimal control problems is of particular interest because of the difficulty in obtaining switching points and optimal solution.

The aim of this study was to present an indirect improved shooting method based on using the Pontryagin's minimum principle for solving this class of switching optimal control problems. Because the switching optimal control problems have discontinuities in the controls and in the derivatives of the states, the conventional shooting method in solving them can cause the accuracy of method, especially in finding the switching points, is deteriorated. Therefore, to cope with this deficiency, an improved shooting method with a control parameterization is presented. For this purpose, at first, based on the Pontryagin's minimum principle, the first order necessary conditions of optimality, which lead to the Hamiltonian boundary value problem (HBVP) are derived. Then, with a knowledge about the number of the switching points and structure of the optimal control obtained from the conventional shooting method, control functions are replaced with piecewise constant functions. Consequently, the problem is converted to the solution of the "shooting equations", in which the values of the switching points and the initial values of the costate variables are unknown parameters.

2. Problem Statement

We consider the switching optimal control problems with control appearing linearly. Suppose, $\mathbf{x}(t)=[x_1(t),...,x_p(t)]^T$ denotes the state vector and $\mathbf{u}(t)=[u_1(t),...,u_q(t)]^T$ is the control vector in the time interval $t\in[0,t_f]$, with a free final time t_f which minimizes

$$J=\int_0^{t_f}1dt=t_f,$$

subject to the constraints on the interval $[0,t_f]$

$$\begin{aligned}\dot{\mathbf{x}}(t)=\mathbf{h}(\mathbf{x}(t),\mathbf{u}(t),t)&=f_1(\mathbf{x}(t),t)+f_2(\mathbf{x}(t),t)\mathbf{u}(t),\\ \mathbf{x}(0)&=\mathbf{x}_0,\\ \mathbf{x}(t_f)&=\mathbf{x}_f,\\ u_j(t)\in\{-1,+1\},&\ \ j=1,2,\dots,q.\end{aligned}$$

Here, the state $\mathbf{x}$ is continuous and controls have finite number of switching points in which controls jump from one value to another at these points. The function $\mathbf{h}:\mathbb{R}^{p+q+1}\to\mathbb{R}^p$ is also assumed to be a smooth function of the variables $(\mathbf{x},\mathbf{u},t)$.

3. The Proposed Method

To formulate the first order necessary conditions of optimality, the Hamiltonian function is introduced as follows

$$\begin{aligned}\mathcal{H}(\mathbf{x}(t),\lambda(t),\mathbf{u}(t))&=1+\lambda^T(t)\mathbf{h}(\mathbf{x}(t),\mathbf{u}(t),t)\\ &=1+\lambda^T(t)f_1(\mathbf{x}(t),t)+\lambda^T(t)f_2(\mathbf{x}(t),t)\mathbf{u}(t),\end{aligned}$$

where, $\lambda(t)=[\lambda_1(t),\dots,\lambda_p(t)]^T$ is called the costate variables. Also, the factor of the control $\mathbf{u}(t)$ in the Hamiltonian function is called the switching function and denoted by

$$\sigma(\mathbf{x},\lambda,t)=[\sigma_1(\mathbf{x},\lambda,t),\dots,\sigma_q(\mathbf{x},\lambda,t)]=\lambda^T f_2(\mathbf{x},t).$$

It is noted that the solution of the optimal control problem satisfies the following necessary conditions

$$\begin{aligned}\dot{\mathbf{x}}&=[\partial\mathcal{H}/\partial\lambda]^T,\\ \dot{\lambda}&=-[\partial\mathcal{H}/\partial\mathbf{x}]^T,\end{aligned}$$

and based on the Pontryagin's minimum principle, an optimal control must minimize the Hamiltonian function with related to the control functions, which readily implies the following control law

$$u_j(t)=\begin{cases}-1, & \text{if } \sigma_{\mathrm{j}}(\mathbf{x},\lambda,\mathrm{t})>0,\\ +1, & \text{if } \sigma_{\mathrm{j}}(\mathbf{x},\lambda,\mathrm{t})<0,\\ \text{undefined}, & \text{if } \sigma_{\mathrm{j}}(\mathbf{x},\lambda,\mathrm{t})=0,\end{cases}$$

for $j=1,2,\dots,q$. So, the switching function $\sigma(t)$ determines the optimal control via

$$\mathbf{u}(t)=-\mathrm{sign}(\sigma(t)),$$

on $[0,t_f]$, except on the switching points. It is noted that, when the final time t_f is free, then the condition

$$\mathcal{H}(\mathbf{x}(t_f),\lambda(t_f),\mathbf{u}(t_f))=0,$$

as transversality condition, must be considered beside the initial and boundary conditions of the problem. Now, by replacing the control function $\mathbf{u}(t)=-\mathrm{sign}(\sigma(t))$ in the dynamic equations, a HBVP as the necessary optimality conditions is formed. Finally, the resulted HBVP is represented by

$$\begin{cases}\dot{\mathbf{y}}=\mathbf{f}(t,\mathbf{y}), & 0\le t\le t_f\\ \mathbf{y}(0)=\mathbf{y}_0,\\ \Psi(\mathbf{y}(t_f),t_f)=\mathbf{0},\end{cases}$$

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where, $\mathbf{y}(t) = [\mathbf{x}(t), \lambda(t)]^T \in \mathbb{R}^{2p}$, $\mathbf{f} : \mathbb{R}^{2p+1} \to \mathbb{R}^{2p}$, and $\Psi : \mathbb{R}^{2p+1} \to \mathbb{R}^{p+1}$ is the boundary function of the problem. Here, the problem is to determine a vector $\mathbf{v} = (\lambda_1(0), \ldots, \lambda_p(0), t_f)^T$, such that, the shooting equations

$$\mathbf{F}(\mathbf{v}) = \Psi(\mathbf{y}(t_f), t_f) = \begin{pmatrix} \mathbf{x}(t_f) - \mathbf{x}_f \\ \mathcal{H}(\mathbf{x}(t_f), \lambda(t_f), -\mathrm{sign}(\sigma(t_f))) \end{pmatrix} = \mathbf{0},$$

is satisfied. It is noted that, because of the existence of discontinuous integrated function in right-hand-side of the above system of differential equations, solving the shooting equations can cause the accuracy of the solution of the HBVP especially in finding the switching points, is deteriorated. The next section is devoted to the efficient and accurate numerical solution of this HBVP.

An Improved Shooting Method with a Control Parameterization

In this section, with a knowledge about the number of the switching points and structure of the optimal control obtained from the conventional shooting method explained in the previous section, suppose that each of control function $u_j(t)$ has n_j switching points. So, the following approximation for each control $u_j(t)$, $j = 1, 2, \ldots, q$,

$$u_j(t) = \begin{cases} b_j^0, & s_j^0 \le t \le s_j^1, \\ b_j^1, & s_j^1 < t \le s_j^2, \\ \vdots & \\ b_j^{n_j}, & s_j^{n_j} < t \le s_j^{n_j+1}, \end{cases}$$

is intended, where $0 = s_j^0 < s_j^1 < \cdots < s_j^{n_j} < s_j^{n_j+1} = t_f$, are considered as unknown switching and terminal points, in which on each subdomain, $u_j(t)$ takes known parameter b_j^k for $k = 0, 1, \ldots, n_j$. So, given that, $b_j^k \in \{-1, +1\}$, where, $b_j^{k+1} = -b_j^k$, consequently we have $b_j^k = -\mathrm{sign}(\sigma_j(t))$, $s_j^k < t \le s_j^{k+1}$, and $b_j^{n_j} = (-1)^{n_j} b_j^0$. It is noted that, according to such approximation for each of control $u_j(t)$, we parameterized it by $\mathbf{s}_j = (s_j^1, \ldots, s_j^{n_j}, t_f)$. Now, by replacing the control function $\mathbf{u}(t; \mathbf{s}) = [u_1(t; \mathbf{s}_1), \ldots, u_q(t; \mathbf{s}_q)]^T$, in the dynamic equations of the problem, the new HBVP is deduced as

$$\begin{cases} \dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}, \mathbf{u}(t; \mathbf{s})), \quad 0 \le t \le t_f \\ \mathbf{y}(0) = \mathbf{y}_0, \\ \Psi(\mathbf{y}(t_f), \mathbf{s}_1, \ldots, \mathbf{s}_q, t_f) = \mathbf{0}. \end{cases}$$

So, the new shooting equations is constructed by

$$\Psi(\cdot) = \begin{pmatrix} \mathbf{x}(t_f) - \mathbf{x}_f \\ \mathcal{H}(\mathbf{x}(t_f), \lambda(t_f), -\mathrm{sign}(\sigma(t_f))) \\ \sigma_j(s_j^1) \\ \vdots \\ \sigma_j(s_j^{n_j}) \end{pmatrix}, \quad j = 1, 2, \ldots, q,$$

where, $\Psi(\cdot)$ is used instead of $\Psi(\mathbf{y}(t_f), \mathbf{s}_1, \ldots, \mathbf{s}_q, t_f)$ for simplicity in notation. The purpose is to solve the shooting equations $\mathbf{S}(\mathbf{z}) = \Psi(\cdot) = \mathbf{0}$, in which, the column vector $\mathbf{z}$, is a vector of unknown parameters and considered as $\mathbf{z} = (\mathbf{s}, \lambda_1(0), \ldots, \lambda_p(0))^T$, where $\mathbf{s} = \left(\mathbf{s}_1, \ldots, \mathbf{s}_q\right)$.

4. Conclusion

An improved shooting method with control parameterization was applied in this paper for the indirect solution of a class of switching optimal control problems. The present method satisfies the first order necessary conditions of optimality and it can capture the switching points accurately. Five illustrative examples were given to demonstrate the validity and applicability of the proposed method.