Journal of Applied and Computational Sciences in Mechanics, Vol. 30, No. 1, 2019.

Verification of Quarter-step Exponential Map Method for Integration of Von-Mises Plasticity

N. Haji Aghajanpour[1], M. Sharifian[2], M. Sharifian[3]

1- Introduction

Updating stress in a nonlinear finite element analysis is the most important part as the precision of the stress-updating algorithm greatly affects the accuracy of the final solutions; the most important part of the analysis is the stress-updating. There are two key factors that have an impact on the efficiency evaluation; i.e. the accuracy and time. Based on this point, the investigation of the accuracy of the integration methods becomes important. The methods used for integrating plasticity differential equations can be divided into three-major groups including explicit, implicit and accurate methods. It should be noted that, exponential map methods which are in the explicit group, the integration operates in an augmented stress space.

Due to lots of calculations in the nonlinear analysis, choosing the more accurate method can reduce the analysis time dramatically. It means one can reach the intended accuracy in less time.

The exponential map methods are more accurate than the other methods mentioned above. In addition, most of them are compatible with the yield surface. It is worth saying, the explicit nature of the exponential map methods causes the high efficiency of them. Due to the aforementioned features of the exponential map methods, the writers of this paper were persuaded to conduct this research.

As mentioned before, the accuracy of integration methods greatly influences their efficiency. Moreover, in the semi-implicit exponential map methods, the common approach is to pick up the variables from the middle of the plasticity step. In this study, the purpose is to evaluate the effects of choosing the variables at the first quarter of the plasticity step on the accuracy and the convergence rate to the accurate response. In order to reach this target, a von-Mises plasticity is taken into account with linear isotropic and kinematic hardening. The relationships of the exponential map method are constructed so that one can pick up the variables from the first quarter of the plasticity step. Finally, a number of numerical tests are performed to determine the accuracy and compare the methods.

2- Basic model

A von-Mises plasticity model with linear isotropic and kinematic hardening in small strain realm is considered. The most important equations related to this model are as follows:

$$F = \|\boldsymbol{\Sigma}\| - R = 0 \tag{1}$$
$$R = R_0 + H_{\mathrm{iso}}\gamma \tag{2}$$
$$\dot{\boldsymbol{\alpha}} = H_{\mathrm{kin}}\dot{\mathbf{e}}^{\mathrm{p}} \tag{3}$$
$$\dot{\gamma} \geq 0 \ , \ F \leq 0 \ , \ \dot{\gamma}F = 0 \tag{4}$$

The above relations indicate the von-Mises yield surface, the isotropic linear hardening rule, the Prager's kinematic hardening rule and the Kuhn-Tucker loading-unloading conditions, respectively.

3- Augmented differential equations system

The constitutive equations related to the von-Mises plasticity model with linear mixed hardening can be converted into the following nonlinear differential equation system using the definition of non-dimensional shifted stress vector ($\bar{\boldsymbol{\Sigma}}$) and the integration factor X^0 as a function of γ:

$$\dot{\mathbf{X}} = \mathbb{A}\mathbf{X} \tag{5}$$

where, $\mathbf{X}$ is the augmented stress vector with the dimension $n+1$:

$$\mathbf{X} = \begin{Bmatrix} X^0\bar{\boldsymbol{\Sigma}} \\ X^0 \end{Bmatrix} = \begin{Bmatrix} \mathbf{X}^s \\ X^0 \end{Bmatrix} \tag{6}$$

The $\mathbb{A}$ matrix for each elastic and plastic part has different definitions:

$$\mathbb{A}_{\mathbf{e}} = \frac{2G}{R}\begin{bmatrix} \mathbb{O}_{9\times9} & \dot{\mathbf{e}}_{9\times1} \\ \mathbf{0}_{1\times9} & 0 \end{bmatrix}_{10\times10} \tag{7}$$
$$\mathbb{A}_{\mathbf{p}} = \frac{2G}{R}\begin{bmatrix} \mathbb{O}_{9\times9} & \dot{\mathbf{e}}_{9\times1} \\ \dot{\mathbf{e}}^{T}{}_{1\times9} & 0 \end{bmatrix}_{10\times10} \tag{8}$$

4- Numerical algorithm for stress updating

In stress updating process, the variable values at time $t = t_n$ $(\gamma_n, \boldsymbol{\alpha}_n, \mathbf{e}_n, \mathbf{s}_n)$ and the strain vector at time $t = t_{n+1}$ $(\mathbf{e}_{n+1})$ are known. The proposed algorithm should solve the plasticity equations and update the stress and other variables at time $t = t_{n+1}$. Conventionally, the strain rate is considered constant ($\dot{\mathbf{e}} = \mathrm{cte}$) in the selected strain paths for the accuracy investigations.

The following initial condition is used to obtain the solution of the differential equation (5):

$$\mathbf{X}(0) = \begin{Bmatrix} \mathbf{X}_0^s \\ 1 \end{Bmatrix} = \begin{Bmatrix} \frac{\boldsymbol{\Sigma}_0}{R_0} \\ 1 \end{Bmatrix} \tag{9}$$

[1] Department of Civil Eng., Quchan University of Technology, Quchan, Iran
[2] Corresponding Author: Department of Civil Eng., Quchan University of Technology, Quchan, Iran.
m.sharifian@qiet.ac.ir
[3] Department of Civil Eng., Quchan University of Technology, Quchan, Iran

The solution of the differential equation (5) using the initial condition (9) is as follows:

$$\mathbf{X}_{n+1} = \exp(\mathbb{A}.\Delta t)\,\mathbf{X}_n \tag{10}$$

In solving problems through the proposed algorithm, the material status should be checked. In each step of the analysis, if the material becomes plastic, the step is commonly divided into two parts, i.e. fully-elastic and elasto-plastic parts. In the algorithm, this division is done using the parameter $0 \leq \alpha \leq 1$; α is applied for the elastic part and $(1-\alpha)$ for the elasto-plastic part.

5- Numerical Tests

In this paper, firstly, the verification of Quarter-step exponential map method is checked. After that, the accuracy of the proposed method and its convergence rate to the accurate response are investigated. The results are presented in different figures and tables.

In order to verify the Quarter-step exponential map method presented, two specified strain histories are considered. Subsequently, the relative stresses are updated for two different materials. Stresses are updated with very fine time step ($\Delta t = 1 \times 10^{-5}$sec) using the proposed method and the explicit exponential map technique. If the results obtained from the mentioned methods are equal, the validity of the proposed method is concluded.

In addition, a boundary value problem (rectangular plate under tension loading) is solved by the proposed algorithm. The obtained results are compared with the results from the FEAP software. The FEAP is a finite-element software based on Backward-Euler method. In the results presented, the validity of the Quarter-step method is observed clearly.

In order to assess the accuracy of the Quarter-step method, the stresses are computed for the aforementioned strain histories and materials. The stress relative error is calculated using the following relation:

$$E_n^{\sigma} = \frac{\|\sigma_n - \bar{\sigma}_n\|}{\|\bar{\sigma}_n\|} \tag{11}$$

where, $\bar{\sigma}_n$ is the accurate stress vector and σ_n is the updated stress vector at time $t = t_n$. The accuracy of the outcomes achieved from conventional semi-implicit method is more than those from the Quarter-step method. Moreover, the convergence rate to the accurate response is calculated using the average of the relative errors through whole path and is compared with the conventional semi-implicit method. The average is calculated as follows:

$$E_T^{\sigma} = \frac{1}{N}\sum \frac{\|\sigma_n - \bar{\sigma}_n\|}{\|\bar{\sigma}_n\|} \tag{12}$$

The tables presented in the paper indicate that the convergence rate of the Quarter-step method to the accurate response is first order while the convergence rate of the middle-step method is second order.

6- Conclusion

In this paper, an integration algorithm, called Quarter-step exponential map, is presented for von-Mises plasticity with linear isotropic and kinematic hardening. The main target of the present work is to investigate the semi-implicit exponential map method. The framework of the method is constructed so that the variables are taken from the first quarter of the plasticity step. Based on the results obtained from the numerical tests, the following can be concluded:

In the semi-implicit exponential map technique, in which the variables are picked up from the middle of the plasticity step, the accuracy is more than the state in which the variables are chosen at the first quarter of plasticity step. In addition, the convergence rate to the accurate response in the former is second order whereas the latter displays a first-order rate.

The accuracy of integration methods for the plasticity equations is significant. If the accuracy of the method increases, one can use larger time steps for the analysis. Moreover, the achievement of the required accuracy of a problem and reduction of the analysis time occur simultaneously.